Notes on Stochastic Volatility Modelling

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Contents

Preface

A Note to the reader

There is something slightly theatrical about volatility.

Unlike price, which has the decency to reveal itself directly, volatility prefers inference, implication, and occasional misdirection. Entire markets quote and trade an object that nobody can observe, yet everyone insists they understand.

These notes were written in an effort to understand that object more honestly.

They owe much to a period spent in the company of practitioners for whom stochastic volatility was neither purely academic abstraction nor merely a pricing input, but a living geometry connecting derivatives, risk, and market behaviour. Exposure to such thinking leaves permanent side effects, including an unreasonable affection for Itô expansions and a tendency to view implied volatility surfaces with more sympathy than suspicion.

The treatment here is intentionally mechanical and mathematical. Most formulas are derived rather than presented, partly from conviction and partly from distrust. Markets may tolerate hand-waving; notes should not.

The reader will therefore encounter a generous amount of algebra, stochastic calculus, and occasionally the author's inability to leave a derivation unfinished.

About the Author

Kenneth Zhang is a quantitative researcher specializing in stochastic volatility, derivatives, and systematic trading research. His work spans volatility modeling, market microstructure, and machine learning applications in quantitative finance. Previously, he worked as a Quantitative Researcher within Systematic Volatility, where he studied options markets, volatility surfaces, and systematic derivatives strategies under the mentorship and guidance of Lorenzo Bergomi and Charbel Saad.

Kenneth's research interests lie at the intersection of stochastic calculus, volatility modeling, statistical learning, and market microstructure. These notes reflect both academic study and practical experience, with an emphasis on mathematical rigor, full derivations, and developing intuition through first principles rather than black-box results.

1 Black-Scholes PDE

1.1 Deriving the Black-Scholes PDE

For any pricing PDE to be valid, we require that the terminal condition be fulfilled for all S (underlying prices). That is, $P(t,S) = f(S)$ for all S where $f(\cdot)$ denotes the payoff function and $P(t,S)$ denotes the pricing model at time t with corresponding stock price S. More concretely, we require that $P(t=T, S_T) = f(S_T)$ for terminal price S_T. Assuming that we are short the option, then any change incurred over some small time interval δt delivers a negative value of our profit and loss (PnL).

Since we are short the option, we earn interest on the option's premium. That is, we earn $rP(t,S)\delta t$. To hedge against moves in the underlying, we buy Δ shares of the underlying, of which we finance at rate r. Therefore buying + financing the underlying hedging position contributed $\Delta\delta S - r\Delta S\delta t$ over the small time interval δt. We also earn money on dividends on the underlying given by q so the dividend paying component pays $q\Delta S\delta t$.

Thus, our overall PnL can be written as,

$$\begin{aligned}
PnL &= -[P(t+\delta t, S+\delta t) - P(t,S)] + rP(t,S)\delta t + \Delta\delta S - r\Delta S\delta t + q\Delta S\delta t \\
&= -[P(t+\delta t, S+\delta S) - P(t,S)] + rP(t,S)\delta t + \Delta(\delta S - rS\delta t + qS\delta t) \\
&= -[\cancel{P(t,S)} + \underbrace{P_t\delta t + P_S\delta S + \frac{1}{2}P_{SS}(\delta S)^2}_{\text{Taylor Expansion of } P(t+\delta t, S+\delta S)} - \cancel{P(t,S)}] + rP\delta t + \Delta(\delta S - rS\delta t + qS\delta t) \\
&= -P_t\delta t - P_S\delta S - \frac{1}{2}P_{SS}(\delta S)^2 + rP\delta t + \Delta\delta S - r\Delta S\delta t + q\Delta S\delta t \\
&= \cancel{(\Delta - P_S)\delta S} + (-P_t + rP - r\Delta S + q\Delta S)\delta t - \frac{1}{2}P_{SS}(\delta S)^2, \quad \text{choose } \Delta := P_S \\
&= (-P_t + rP - rSP_S + qSP_S)\delta t - \frac{1}{2}P_{SS}(\delta S)^2 \\
&= -(P_t - rP + (r-q)SP_S)\delta t + \frac{1}{2}S^2P_{SS}\left(\frac{\delta S}{S}\right)^2. \qquad \text{(Equation 1.1)}
\end{aligned}$$

The realized variance $\left(\frac{\delta S}{S}\right)^2$ over δt is of order $O(\sqrt{\delta t})$ so $\delta S = O(\sqrt{\delta t})$ meaning that $(\delta S)^2 = O(\delta t)$. Naturally, $\delta t\delta S$ is of order $O(\delta t^{3/2})$ meaning $(\delta t)^2 = O(\delta t)^2$. We keep our PDE of first order, so we only keep terms $P_t\delta$, $P_S\delta S$ and $\frac{1}{2}P_{SS}(\delta S)^2$ which after simplifying yields Equation 1.1.

Suppose we let $A(t,S)$ be the deterministic theta component $(P_t - rP + (r-q)SP_S)$ and let $B(t,S)$ be our gamma component $\frac{1}{2}S^2P_{SS}$. Clearly, if $A, B \geq 0$ then $-A\delta t \leq 0$ and $-Bx \leq 0$ for $x = (\delta S/S)^2$ meaning $PnL \leq 0$. Clearly, our pricing function will consistently underprice any option. Conversely, if $A, B \leq 0$, then $PnL \geq 0$ for all x meaning that our pricing function will consistently overprice all options.

We aim to deduce a implied break-even variance from this equation.

For our pricing function to be deemed usable, and more importantly, fair, we require that the breakeven volatility satisfies $PnL = 0$ that is $-A\delta t - B\left(\frac{\delta S}{S}\right)^2$. It follows that such break-even volatility is,

$$\begin{aligned}
0 = -A\delta t - B\left(\frac{\delta S}{S}\right)^2 &\iff -A\delta t = B\left(\frac{\delta S}{S}\right)^2 \\
&\iff \left(\frac{\delta S}{S}\right)^2 = -\frac{A}{B}\delta t \\
&\iff \frac{\delta S}{S} = \pm\sqrt{-\frac{A(t,S)}{B(t,S)}\delta t} \in \mathbb{R} \\
&\iff -\frac{A}{B} > 0.
\end{aligned}$$

It is important to note that we are working over arbitrarily small time intervals δt, so for some realized variance over the time interval $\left(\frac{\delta S}{S}\right)^2$ we estimate $\hat{\sigma}$ as the volatility level over δ. This is an approximation of the realized variance realized over the same interval i.e., $\left(\frac{\delta S}{S}\right)^2 \approxeq \hat{\sigma}^2 \delta t$.

Clearly, the condition that our $PnL = 0$ is satisfied if and only if $\left(\frac{\delta S}{S}\right)^2 \approxeq \hat{\sigma}^2 \delta t$.

Substituting realized volatility with our realized volatility estimate,

$$
\begin{aligned}
PnL = 0 &= -A\delta t - B\left(\frac{\delta S}{S}\right)^2 \\
&= -A\delta t - B\hat{\sigma}^2\delta t \\
&= -(A + B\hat{\sigma}^2)\delta t \\
A &= -B\hat{\sigma}^2.
\end{aligned}
$$

Substituting this in with the definitions of A and B we get,

$$
\begin{aligned}
P_t - rP + (r-q)SP_S &= -\hat{\sigma}^2 \frac{1}{2} S^2 P_{SS} \\
P_t - rP + (r-q)SP_S + \frac{1}{2}\hat{\sigma}^2 S^2 P_{SS} &= 0. \qquad \text{(Equation 1.2)}
\end{aligned}
$$

Equation 1.2 above is the famed *Black-Scholes PDE*.

1.2 Realized vs. Model Implied Break-Even Variance

Recall that we had previously written $A = -B\hat{\sigma}^2$. Substituting this in for A we get,

$$
\begin{aligned}
PnL &= -(-\hat{\sigma}^2\delta t) - B\left(\frac{\delta S}{S}\right)^2 \\
&= \hat{\sigma}^2 B\delta t - B\left(\frac{\delta S}{S}\right)^2 \\
&= -B\left[\left(\frac{\delta S}{S}\right)^2 - \hat{\sigma}^2\delta t\right] \\
&= -\frac{1}{2}S^2 P_{SS}\left\{\left(\frac{\delta S}{S}\right)^2 - \hat{\sigma}^2\delta t\right\}.
\end{aligned}
$$

This is imply the difference between the realized variance minus the model implied break-even variance multiplied by the negative-half dollar-weighted gamma of the option (higher-order delta sensitivity). Since we are short the option, and we usually have $P_{SS} > 0$ and $\frac{1}{2}S^2 P_{SS} > 0$ then we lose money if the realised variance is greater than our implied and gain if the model implied is greater than the realised.

In general, we require that under the risk-neutral measure, the expected discounted future realized variance must equal the expected discounted value of the option's gamma-weighted exposure to instantaneous variance.

$$
\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-rt} S_t^2 \sigma_t^2 dt\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^t e^{-rt} S_t^2 \frac{\partial^2 P}{\partial S^2}\hat{\sigma}^2 dt\right] \iff \left\langle \int_0^T e^{-rt} S_t^2 \frac{\partial^2 P}{\partial S^2}\sigma_t^2 dt \right\rangle = \left\langle \int_0^T e^{-rt} S_t^2 \frac{\partial^2 P}{\partial S^2}\hat{\sigma}^2 dt \right\rangle.
$$

Here, the term $S_t^2 \frac{\partial^2 P}{\partial S^2}$ denotes the option's dollar gamma exposure determining how sensitive the hedge is to quadratic variation / volatility. Therefore, the market's true instantaneous variance process σ_t^2 and option's pricing implied/local volatility $\hat{\sigma}$, under no-arbitrage pricing, requires that the gamma-weighted average variance experienced by the option must match the gamma-weighted implied variance used in pricing.

1.3 Discretized PnL Attribution

In the real world, it is almost impossible for us to carry out continuous delta-hedging. However, suppose that we hedge at time t_i so that we instead discretize our PnL equation to the discrete delta-hedging case.

That is, instead of integrating over the discounted future variance, we write the integral as a discrete approximation of the integral. Suppose that instead we also observe returns r_i such that squaring gives us r_i^2.

Under Black-Scholes assumptions, our returns r_i can be written as $\sigma_i z_i$ so that $r_i^2 = \sigma_i^2 z_i^2$ where z_i's are iid standard normal random variables. Therefore, our PnL can be written as,

$$\begin{aligned} PnL &= -\sum_i e^{-rt_i}\frac{1}{2}S_i^2 P_{SS}^{\hat{\sigma}}(t_i, S_i)(r_i^2 - \hat{\sigma}^2\delta t) \\ &= -\frac{1}{2}S_0^2 P_{SS}^{\hat{\sigma}}\sum_i(\sigma_i^2 z_i^2 - \hat{\sigma}^2)\delta t. \end{aligned}$$

Note that we have wrtten $e^{-rt_i}S_i^2 P_{SS}^{\hat{\sigma}}(t_i, S_i)$ as $S_0^2 P_{SS}^{\hat{\sigma}}$ because we are discounting to time $t = 0$. To be more explicit, we have written $P_{SS}^{\hat{\sigma}}$ to outline the dependence of P_{SS} on $\hat{\sigma}$. $-\frac{1}{2}S_0^2 P_{SS}^{\hat{\sigma}}$ is clearly deterministic.

We wish to make sense of $\sum_i(\sigma_i^2 z_i^2 - \hat{\sigma}^2)$ in our PnL.

Let's first compute the expectation of the sum,

$$\begin{aligned} \left\langle \sum_i(\sigma_i^2 z_i^2 - \hat{\sigma}^2)\right\rangle &= \sum_i \left(\left\langle \sigma_i^2 z_i^2\right\rangle - \hat{\sigma}^2\right)\delta t \\ &= \sum_i \left(\left\langle \sigma_i^2\right\rangle \left\langle z_i^2\right\rangle - \hat{\sigma}^2\right)\delta t \\ &= \sum_i \left(\left\langle \sigma_i^2\right\rangle \cdot 1 - \hat{\sigma}^2\right)\delta t \quad \text{since } \langle z_i\rangle = 1, \\ &= \sum_i(\hat{\sigma}^2 - \hat{\sigma}^2)\delta t \\ &= 0. \end{aligned}$$

Since the expectation of the sum is zero, the variance is simply computed as,

$$\begin{aligned} \left\langle \left[\sum_i(\sigma_i^2 z_i^2 - \hat{\sigma}^2)\delta t\right]^2\right\rangle &= \left\langle \sum_{i,j}(\sigma_i^2 z_i^2 - \hat{\sigma}^2)(\sigma_j^2 z_j^2 - \hat{\sigma}^2)\delta t^2\right\rangle \\ &= \sum_{i,j}\left\langle (\sigma_i^2 z_i^2 - \hat{\sigma}^2)(\sigma_j^2 z_j^2 - \hat{\sigma}^2)\delta t^2\right\rangle. \end{aligned}$$

We will decompose this double sum into $i = j$ (diagonal) components and $i \neq j$ (off-diagonal) components.

Starting with the diagonal components,

$$\begin{aligned} (\sigma_i^2 z_i^2 - \hat{\sigma}^2)(\sigma_i^2 z_i^2 - \hat{\sigma}^2) &= (\sigma_i^2 z_i^2 - \hat{\sigma}^2) \\ &= \sigma_i^4 z_i^4 - 2\hat{\sigma}^2\sigma_i^2 z_i^2 + \hat{\sigma}^4 \\ &\approxeq \left\langle \sigma_i^4 z_i^4\right\rangle - 2\hat{\sigma}\left\langle \sigma_i^2 z_i^2\right\rangle + \hat{\sigma}^4 \\ &\approxeq \left\langle \sigma_i^4 z_i^4\right\rangle - 2\hat{\sigma}^4\hat{\sigma}^4 + \hat{\sigma}^4 \\ &= \left\langle \sigma_i^4 z_i^4\right\rangle - 2\hat{\sigma}^4 + \hat{\sigma}^4 \\ &= \left\langle \sigma_i^4 z_i^4\right\rangle - \hat{\sigma}^4. \end{aligned}$$

We wish to make sense of $\langle \sigma_i^4 z_i^4 \rangle$. From the fourth moment we have that,

$$\mathbb{E}[X^4] = 3\sigma^4.$$

It follows for a standard normal random variable, κ controls the excess kurtosis,

$$\kappa = \frac{\mathbb{E}[r_i^4]}{\sigma^4} - 3 \implies \mathbb{E}[r_i^4] = (3+\kappa)\hat{\sigma}^4.$$

Thus we have that,

$$\left\langle \sigma_i^4 z_i^4 \right\rangle - \hat{\sigma}^4 = (3+\kappa)\hat{\sigma}^4 - \hat{\sigma}^4 = (2+\kappa)\hat{\sigma}^4,$$

for diagonal terms where $i = j$. For off-diagonal terms where $i \neq j$,

$$\begin{aligned}
\left\langle (\sigma_i^2 z_i^2 - \hat{\sigma}^2)(\sigma_j^2 z_j^2 - \hat{\sigma}^2) \right\rangle &= \left\langle \sigma_i^2 \sigma_j^2 z_i^2 z_j^2 - \hat{\sigma}^2 \sigma_i^2 z_j^2 - \hat{\sigma}^2 \sigma_j^2 z_j^2 + \hat{\sigma}^4 \right\rangle \\
&= \left\langle \sigma_i^2 \sigma_j^2 z_i^2 z_j^2 \right\rangle - \hat{\sigma}^2 \left\langle \sigma_i^2 z_i^2 \right\rangle - \hat{\sigma}^2 \left\langle \sigma_j^2 z_j^2 \right\rangle + \hat{\sigma}^4 \\
&= \left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \hat{\sigma}^2 \left\langle \sigma_i^2 z_i^2 \right\rangle - \hat{\sigma}^2 \left\langle \sigma_j^2 z_j^2 \right\rangle + \hat{\sigma}^4 \\
&= \left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \hat{\sigma}^2 \hat{\sigma}^2 - \cancel{\hat{\sigma}^2 \hat{\sigma}^2 + \hat{\sigma}^4} \\
&= \left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \hat{\sigma}^4.
\end{aligned}$$

The expected value of the sum term is decomposed into a diagonal component and an off-diagonal component.

$$\left\langle \left[\sum_i (\sigma_i^2 z_i^2 - \hat{\sigma}^2)^2 \delta t^2 \right] \right\rangle = \sum_i (2+\kappa)\hat{\sigma}^4 \delta t^2 + \sum_{i \neq j} \left(\left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \hat{\sigma}^4 \right) \delta t^2.$$

The second component can be expressed with respect to a normalized auto-correlation term, or a two-time autocovariance term of instantaneous variance. We'll let the normalized form of this be denoted as,

$$f_{ij} = \frac{\left(\left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \hat{\sigma}^4 \right)}{\langle \sigma^4 \rangle - \hat{\sigma}^4}.$$

Thus, we write the term now as,

$$\begin{aligned}
\left\langle \left[\sum_i (\sigma_i^2 z_i^2 - \hat{\sigma}^2)^2 \delta t^2 \right] \right\rangle &= \sum_i (2+\kappa)\hat{\sigma}^4 \delta t^2 + \sum_{i \neq j} \left(\left\langle \sigma^4 \right\rangle - \hat{\sigma}^4 \right) \cdot f_{ij} \cdot \delta t^2 \\
&= \sum_i (2+\kappa)\hat{\sigma}^4 \delta t + \left(\left\langle \sigma^4 \right\rangle - \hat{\sigma}^4 \right) \sum_{i \neq j} f_{ij} \delta t^2 \\
&= \sum_i (2+\kappa)\hat{\sigma}^4 \delta t^2 + \Omega \hat{\sigma}^4 \sum_{i \neq j} f_{ij} \delta t^2, \quad \text{for } \Omega = \frac{\langle \sigma^4 \rangle - \hat{\sigma}^4}{\hat{\sigma}^4}, \\
&= \hat{\sigma}^4 \left\{ \sum_i (2+\kappa)\delta t^2 + \Omega \sum_{i \neq j} f_{ij} \delta t^2 \right\}.
\end{aligned}$$

Therefore, the variance of our PnL can be expressed as,

$$\begin{aligned}
\mathrm{Var}(PnL) &= \mathrm{Var}\left(-\frac{1}{2} S_0^2 P_{SS}^{\hat{\sigma}}(t_0, S_0) \right)^2 \left\langle \sum_i (\sigma_i^2 z_i^2 - \hat{\sigma}^2) \delta t^2 \right\rangle \\
&= \left[\frac{1}{2} S_0^2 P_{SS}^{\hat{\sigma}}(t_0, S_0) \right]^2 \cdot \hat{\sigma}^4 \left\{ \sum_i (2+\kappa)\delta t^2 + \Omega \sum_{i \neq j} f_{ij} \delta t^2 \right\}.
\end{aligned}$$

Before we compute the standard deviation, I will highlight an important expression. Recall that we can express our option's vega approximately as,

$$\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} = S^2 \frac{\partial^2 P_{\hat{\sigma}}}{\partial S^2} \hat{\sigma} T.$$

Then it follows that we can write the following term as,

$$S^2 \frac{\partial^2 P_{\hat{\sigma}}}{\partial S^2} = \frac{1}{\hat{\sigma} T} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} \implies \frac{1}{2} \frac{1}{\hat{\sigma} T} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} = \frac{1}{2} S^2 \frac{\partial^2 P_{\hat{\sigma}}}{\partial S^2}.$$

Therefore, the standard deviation of our PnL can be computed as,

$$\begin{aligned}
SD(PnL) &= \sqrt{\text{Var}(PnL)} \\
&= \sqrt{\left\{\frac{1}{2} S_0^2 P_{SS}^{\hat{\sigma}}(t_0, S_0)\right\}^2 \cdot \hat{\sigma}^4 \left\{\sum_i (2+\kappa_i)\delta t^2 + \Omega \sum_{i \neq j} f_{ij} \delta t^2\right\}} \\
&= \left|\frac{S_0^2}{2} P_{SS}^{\hat{\sigma}}(t_0, S_0)\right| \cdot \sqrt{\hat{\sigma}^4 \left\{\sum_i (2+\kappa_i)\delta t^2 + \Omega \sum_{i \neq j} f_{ij} \delta t^2\right\}} \\
&= \left|\frac{S_0^2}{2} P_{SS}^{\hat{\sigma}}(t_0, S_0)\right| \cdot \hat{\sigma}^2 \sqrt{\sum_i (2+\kappa_i)\delta t^2}, \quad \text{assuming uncorrelated increments,} \\
&= \left|\frac{S_0^2}{2} P_{SS}^{\hat{\sigma}}(t_0, S_0)\right| \cdot \hat{\sigma}^2 \sqrt{2 \sum_i \delta t^2}, \quad \text{assuming Normality so, } \kappa_i = 0, \\
&= \frac{1}{2} \cdot \left|S_0^2 P_{SS}^{\hat{\sigma}}(t_0, S_0)\right| \hat{\sigma}^2 \cdot \sqrt{2 \sum_i \delta t^2}, \\
&= \frac{1}{2} \left|\frac{1}{\hat{\sigma} T} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \hat{\sigma}^2 \cdot \sqrt{2 \sum_i \delta t^2} \\
&= \frac{1}{2T} \left|\hat{\sigma} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \sqrt{2 \sum_i \delta t^2} \\
&= \frac{1}{2T} \left|\hat{\sigma} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \sqrt{2 \cdot \frac{T^2}{N}} \\
&= \left|\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \frac{\hat{\sigma}}{\sqrt{2N}}.
\end{aligned}$$

Thus, a standard deviation of our PnL is equivalent to the impact of a relative volatility perturbation of size $\frac{1}{\sqrt{2N}}$. This follows because, if we view it as the option price impact caused by a volatility perturbation of size $\delta\hat{\sigma} = \frac{\hat{\sigma}}{\sqrt{2N}}$, then $\frac{\delta\hat{\sigma}}{\hat{\sigma}} = \frac{1}{\sqrt{2N}}$. To see this more clearly, we often use a standard realized variance estimator to estimate the volatility since $\hat{\sigma}^2$ is usually not available to us. We often use the estimator,

$$\bar{\sigma}^2 = \frac{1}{N} \sum_i \left(\frac{S_{i+1} - S_i}{S_i}\right)^2 \approx \frac{1}{N} \hat{\sigma}^2 \sum_i z_i^2.$$

In Black-Scholes, we assume that the return is approximately Gaussian, that is $r_i \approx \hat{\sigma} z_i$ where $z_i \sim \mathcal{N}(0,1)$.

Computing the expectation we get,

$$\mathbb{E}[\bar{\sigma}^2] \approx \mathbb{E}\left[\frac{\hat{\sigma}^2}{N} \sum_i z_i^2\right] = \frac{\hat{\sigma}^2}{N} \sum_i \mathbb{E}[z_i^2] = \frac{\hat{\sigma}^2}{N} \sum_i 1 = \hat{\sigma}^2.$$

It is clear that this estimator is centered around the true variance and is therefore unbiased. We make not statements on whether this is the best unbiased estimator for realized variance.

Now, we may compute the variance,

$$\mathrm{Var}(\bar{\sigma}^2) \approx \mathrm{Var}\left(\frac{\bar{\sigma}^2}{N}\sum_i z_i^2\right) = \frac{\hat{\sigma}^2}{N^2}\mathrm{Var}\left(\sum_i z_i^2\right).$$

To find the variance of the independent z_i's, we use the 4th and 2 moments of z_i. Recall that for a standard normal variable z_i, $\mathbb{E}[z_i^4] = 3$ and $\mathbb{E}[z_i^2] = 1$. So, trivially $\mathrm{Var}(z_i^2) = \mathbb{E}[z_i^4] - \mathbb{E}[z_i^2]^2 = 3 - 1 = 2$.

Therefore the variance of our estimator is simply,

$$\mathrm{Var}(\bar{\sigma}^2) \approx \frac{\hat{\sigma}^4}{N^2}(2N) = \frac{2\hat{\sigma}^4}{N}.$$

Taking the root of our variance formulation above, we get the standard deviation of our estimator,

$$SD(\bar{\sigma}^2) = \hat{\sigma}^2\sqrt{\frac{2}{N}} = \frac{1}{2}\frac{SD(\bar{\sigma}^2)}{\mathbb{E}[\bar{\sigma}^2]} = \frac{1}{2}\cdot\sqrt{\frac{2}{N}} = \frac{1}{\sqrt{2N}}.$$

However, recall that volatility is the square root of variance so for small errors, if we let $V = \sigma^2$ then $\sigma = \sqrt{V}$. Thus a small change in $\frac{\delta\sigma}{\sigma} \approx \frac{1}{2}\frac{\delta V}{V}$. Thus, a relative standard deviation of the volatility estimator is approximately half the relative standard deviation of the variance estimator,

$$\frac{SD(\bar{\sigma})}{\hat{\sigma}} \approx \frac{1}{2}\cdot\frac{SD(\bar{\sigma}^2)}{\mathbb{E}[\bar{\sigma}^2]} = \frac{1}{2}\sqrt{\frac{2}{N}} = \frac{1}{\sqrt{2N}}.$$

Since we estimated the historical volatility then $SD(\hat{\sigma}_{\text{hist}}) \approx \frac{\hat{\sigma}}{\sqrt{2N}}$ which means a standard deviation in our PnL is equivalent to our option's vega times a perturbation in the volatility,

$$SD(PnL) = \frac{\hat{\sigma}}{\sqrt{2N}}\cdot\left|\frac{\partial P}{\partial\hat{\sigma}}\right| = \frac{1}{\sqrt{2N}}\left|\hat{\sigma}\cdot\frac{\partial P}{\partial\hat{\sigma}}\right|,$$

as we derived previously.

1.4 Pricing an ATM Call Option

Suppose we have a call with $S = 1, K = 1$ and time to maturity $T = 1$ with $r = q = 0$ and $\hat{\sigma} = 20\% = 0.20$.

Recall that the price of a European Call option is,

$$P = N(d_1) - Ke^{-rt}N(d_2) = N(d_1) - 1\cdot e^{-rt}N(d_2).$$

Solving for d_1 and d_2 explicitly we get,

$$d_1 = \frac{\frac{1}{2}\hat{\sigma}^2 T}{\hat{\sigma}\sqrt{T}} = \frac{1}{\hat{\sigma}\sqrt{T}}\cdot\frac{1}{2}\hat{\sigma}\cdot T = \frac{1}{2}\hat{\sigma}\sqrt{T} = 0.10, \qquad d_2 = 0.10 - 0.20 = -0.10.$$

Then, under constant volatility assumptions the Black-Scholes European call option premium is,

$$P = N(0.10) - N(-0.10) = N(0.10) - (1 - N(0.10)) = 2N(0.10) - 1 = 2\cdot 0.54 - 1 \approxeq 0.0796.$$

Under Black-Scholes assumptions, the option's premium is approximately 7.97% of the spot price.

Before we price the same European call option under stochastic volatility assumptions, we will make a claim of which I will prove here. We claim that $P \approxeq \hat{\sigma}\frac{\partial P}{\partial\hat{\sigma}}$ under short-maturity assumptions.

Proof. We claim that,

$$P_{\hat{\sigma}} \approxeq \frac{1}{2} S\hat{\sigma}\sqrt{T}, \qquad \text{(Bergomi 2016.)}$$

Differentiating this with respect to $\hat{\sigma}$ we get,

$$\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} \approxeq \frac{\partial}{\partial \hat{\sigma}} \left(\frac{1}{\sqrt{2\pi}} S\hat{\sigma}\sqrt{T} \right) = \frac{1}{\sqrt{2\pi}} S\sqrt{T}.$$

Multiplying the identity above by $\hat{\sigma}$ we get,

$$\hat{\sigma} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} \approxeq \hat{\sigma} \left(\frac{1}{\sqrt{2\pi}} S\sqrt{T} \right) = \frac{1}{\sqrt{2\pi}} S\hat{\sigma}\sqrt{T} \approxeq P_{\hat{\sigma}}.$$

Thus for an ATM option, the approximation holds because the ATM option price is linear in volatility. □

Thus if we plug in $S = 1$ and $\hat{\sigma} = 0.20$ with $T = 1$,

$$P_{\hat{\sigma}} \approx \frac{1}{\sqrt{2\pi}} \cdot 1 \cdot 0.20 \cdot 1 = \frac{0.20}{\sqrt{2\pi}} \approx 0.0798.$$

This is very close to the exact Black-Scholes price of 7.97%.

What does the standard deviation of our PnL have to say about this price? If we plug in our values,

$$SD(PnL) = \frac{1}{\sqrt{2N}} \left| \hat{\sigma} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} \right| = \frac{1}{\sqrt{2 \cdot 250}} \cdot \left| \hat{\sigma} \frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}} \right| \approx 0.045P.$$

Given our option premium is 7.97% of the spot price, the dollar relative to the spot price is $0.045P \approxeq 0.045 \times 0.0797 = 0.00359$. Therefore, the delta-hedged standard deviation of PnL is 0.36% of the spot. Relative to the premium, $\frac{SD(PnL)}{P} = 0.045$, that is 4.5% of the spot premium or approximately 5%.

For an ATM option, we also are familiar with the approximation that $P \approx C\hat{\sigma}$ where $C = \frac{1}{\sqrt{2\pi}} S\sqrt{T}$ meaning that a change in the option price is approximated by a change in volatility. So, a 4.5% relative uncertainty in volatility produces approximately a 5% relative uncertainty in the option price.

The statistical uncertainty of realized volatility is therefore,

$$\frac{SD(\hat{\sigma}_{\text{realized}})}{\hat{\sigma}} \approx \frac{1}{\sqrt{2N}} \approx 4.5\%.$$

Because an ATM option price is nearly linear in volatility we approximate,

$$\frac{SD(PnL)}{P} \approx \frac{SD(\hat{\sigma}_{\text{realized}})}{\hat{\sigma}} \approx 4.5\%.$$

Suppose that we have a fair option price of P and the risk from the uncertainty in realized volatility is $SD(PnL) \approx 0.05P$. If a dealer wants to change one standard deviation of protection on the offer, the ask price is approximately,

$$P_{\text{ask}} = P + 0.05P = 1.05P.$$

Alternatively, if the dealer wants a one standard deviation of protection on the bid, the bid price is approximately,

$$P_{\text{bid}} = P - 0.05P = 0.95P.$$

Therefore the bid/ask spread is,

$$P_{\text{ask}} - P_{\text{bid}} = 1.05P - 0.95P = 0.10P,$$

therefore the relative bid/ask spread is 10%.

1.5 Non-Black-Scholes PnL

Recall our formulation of the standard deviation of the PnL,

$$SD(PnL) = \left|\hat{\sigma}\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \frac{1}{2T}\sqrt{\sum_i (2+\kappa)\delta t^2 + \Omega \sum_{i \neq j} f_{ij}\delta t^2}.$$

The term involving the excess kurtosis of our Gaussian returns perturbed by a Gaussian shock z_i is random. So, even if daily variances were constant, the realized squared return remains random due to the z_i's with κ adjusting for fatter/thinner tails. The second term, however, is the true stochastic volatility term. Empirically, daily returns are not independent, that is, if today's variance is high, tomorrow's variance is likely to be high too. So, the daily gamma/theta PnLs are correlated across time. Recall,

$$\Omega = \frac{\langle \sigma^4 \rangle - \hat{\sigma}^4}{\hat{\sigma}^4}, \qquad f_{ij} = \frac{\left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \hat{\sigma}^4}{\langle \sigma^4 \rangle - \hat{\sigma}^4},$$

where Ω measures dispersion of daily variances and f_{ij} measures correlation between variance at time i and variance at time j. We approximate the entire sum under time-homogeneous assumptions i.e., $\text{Cov}(\sigma_i^2, \sigma_j^2)$ only depends on time distance, as an integral,

$$\sum_{i \neq j} f_{ij}\delta t^2 \approxeq \int_0^T \int_0^T f(|t-u|)dtdu.$$

Note that the diagonals where $i = j$ have measure zero, therefore,

$$\begin{aligned}
\int_0^T \int_0^T f(|t-u|)dtdu &= 2\int_0^T \int_0^t f(t-u)dudt, \quad \text{symmetric about } [0,T]\times[0,T], \\
&= 2\int_0^T \int_0^t f(\tau)(-d\tau), \quad \tau = t-u, d\tau = -du, \\
&= 2\int_0^T \int_0^t f(\tau)d\tau dt, \quad 0 \leq \tau \leq t \leq T \iff 0 \leq \tau \leq t \wedge \tau \leq t \leq T \\
&= 2\int_0^T \int_\tau^T f(\tau)dtd\tau \\
&= 2\int_0^T f(\tau)(T-\tau)d\tau.
\end{aligned}$$

Moreover since we are hedging N times over δt then $\delta = \frac{T}{N}$ so the first κ term becomes,

$$\sum_i (2+\kappa)\delta t^2 = (2+\kappa)\sum_i \delta t^2 = (2+\kappa)N\delta t^2 = (2+\kappa)N\left(\frac{T}{N}\right)^2 = (2+\kappa)\frac{T^2}{N}.$$

Thus our standard deviation formula of the PnL becomes,

$$\begin{aligned}
SD(PnL) &= \left|\hat{\sigma}\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \frac{1}{2T}\sqrt{\sum_i (2+\kappa)\delta t^2 + \Omega\sum_{i\neq j} f_{ij}\delta t^2} \\
&\approxeq \left|\hat{\sigma}\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \frac{1}{2T}\sqrt{(2+\kappa)\frac{T^2}{N} + 2\Omega\int_0^T (T-\tau)f(\tau)d\tau} \\
&= \left|\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right|\sqrt{\frac{(2+\kappa)T^2/N}{4T^2} + \frac{2\Omega}{4T^2}\int_0^T (T-\tau)f(\tau)d\tau} \\
&= \left|\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right|\sqrt{\frac{2+\kappa}{4N} + \frac{\Omega}{2T^2}\int_0^T (T-\tau)f(\tau)d\tau}.
\end{aligned}$$

From the formulation above, we are clearly able to recover the Black-Scholes PnL standard deviation equation,

$$\Omega = 0, \kappa = 0 \implies SD(PnL) = \left|\hat{\sigma}\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right| \cdot \sqrt{\frac{2}{4N}} = \frac{1}{\sqrt{2N}}\left|\hat{\sigma}\frac{\partial P_{\hat{\sigma}}}{\partial \hat{\sigma}}\right|.$$

In the case we have perfectly correlated daily variances i.e., where $f(\tau) = 1$ then,

$$\int_0^T (T-\tau)f(\tau)d\tau = \int_0^T (T-\tau)d\tau = \left[T\tau - \frac{\tau^2}{2}\right]_0^T = T^2 - \frac{T^2}{2} = \frac{T^2}{2}.$$

Therefore the stockastic volatility term under the square root becomes,

$$\frac{\Omega}{2T^2}\int_0^T (T-\tau)f(\tau)d\tau = \frac{\Omega}{2T^2} \cdot \frac{T^2}{2} = \frac{\Omega}{4}.$$

This term clearly does not vanish as the number of times we hedge goes to infinity. Therefore, more frequent delta-hedging does not remove volatility risk as $N \to \infty$. If only the second term mattered then,

$$SD(PnL) \approx \left|\hat{\sigma}\frac{\partial P}{\partial \hat{\sigma}}\right| \cdot \frac{\sqrt{\Omega}}{2}.$$

Since we know that an change in the option premium is approximately equivalent to a change in the option price with respect to a change in the volatility, then this is equivalent to a volatility perturbation of size,

$$\delta P \approx \frac{\partial P}{\partial \hat{\sigma}}\delta\hat{\sigma} \implies \delta\hat{\sigma} = \hat{\sigma}\frac{\sqrt{\Omega}}{2}.$$

This contribution is equivalent to mispricing $\hat{\sigma}$ by $\hat{\sigma}\frac{\sqrt{\Omega}}{2}$.

Daily returns in the real world are often not enough to cleanly separate into a variance component σ_i and a random shock piece z_i. Thus, we need to estimate daily volatility using intraday data. The daily variance is approximately $\sigma_i^2 = \sum_{\text{5-min intervals in day i}} r_{i,m}^2 + r_{\text{close-to-open, i}}^2$, from which we can comptue the autocorrelation of daily variances. Empirically, daily variances are fit with an exponential,

$$f(\tau) = \rho e^{-k\tau}.$$

Clearly, $\rho < 1$ meaning that even at very short lages, the measured variance autocorrelation is below one. Using this empirical assumption, we can derive the explicit real-world formulation of the standard deviation of the PnL after removing Black-Scholes assumptions. Using our integral approximation we have,

$$\begin{aligned}
\int_0^T (T-\tau)f(\tau)\,d\tau &= \rho\left[T\frac{1-e^{-kT}}{k} - \int_0^T \tau e^{-k\tau}\,d\tau\right] \\
&= \rho\left[T\frac{1-e^{-kT}}{k} - \left(-\frac{T}{k}e^{-kT} + \frac{1-e^{-kT}}{k^2}\right)\right] \\
&= \rho\left[\frac{T}{k}(1-e^{-kT}) + \frac{T}{k}e^{-kT} - \frac{1-e^{-kT}}{k^2}\right] \\
&= \rho\left[\frac{T}{k} - \frac{T}{k}e^{-kT} + \frac{T}{k}e^{-kT} - \frac{1-e^{-kT}}{k^2}\right] \\
&= \rho\left[\frac{T}{k} - \frac{1-e^{-kT}}{k^2}\right] \\
&= \rho\left[\frac{kT}{k^2} - \frac{1-e^{-kT}}{k^2}\right] \\
&= \rho\left[\frac{kT-(1-e^{-kT})}{k^2}\right] \\
&= \rho\left[\frac{kT-1+e^{-kT}}{k^2}\right].
\end{aligned}$$

Plugging this into our vega-normalied standard deviation formulation of our PnL we have,

$$
\begin{aligned}
\frac{SD(PnL)}{\left|\hat{\sigma}\frac{\partial P}{\partial \hat{\sigma}}\right|} &= \sqrt{\frac{2+\kappa}{4N} + \frac{\Omega}{2T^2}\rho\frac{kT-1+e^{-kT}}{k^2}} \\
&= \sqrt{\frac{2+\kappa}{4N} + \frac{\rho\Omega}{2}\frac{kT-1+e^{-kT}}{k^2T^2}} \\
&= \sqrt{\frac{2+\kappa}{4N} + \frac{\rho\Omega}{2}\frac{kT-1+e^{-kT}}{(kT)^2}}.
\end{aligned}
$$

We see that this quantity is the relative volatility displacement that would produce the same option price change as one standard deviation of final delta-hedged PnL specifically from the term $\frac{\rho\Omega}{2}\frac{kT-1+e^{-kT}}{(kT)^2}$.

Using the stochastic volatility model of the standard deviation of PnL we can compute the same ATM European call option price. We'll use the estimates $\Omega = 2, \kappa = 5, \rho = 0.78$ and $\frac{1}{k} = 45$ days.

For a one-year option, there are approximately $N = 250$ trading days meaning that $T = 250$ days. Then $kT = \frac{250}{45} \approx 5.56$ so the dispersion term under the square root becomes,

$$
\frac{2+\kappa}{4N} = \frac{2+5}{(4\times 250)} = \frac{7}{1000} = 0.007.
$$

The stochastic volatility of variances becomes,

$$
\frac{\rho\Omega}{2}\frac{kT-1+e^{-kT}}{(kT)^2} = \frac{0.78\cdot 2}{2}\cdot\frac{5.56-1+e^{-5.56}}{(5.56)^2} \approx 0.78\cdot\frac{4.56}{30.9} \approx 0.115.
$$

The total inside the square root becomes 0.122 so $\sqrt{0.122} \approx 0.35$. So under stochastic volatility, since for an ATM option the option price is linear in volatility, a standard deviation of the PnL is equivalently 35% of the option premium. Black-Scholes showed that under the same ATM European call option, the PnL risk is only about 4.5% of the premium. Stochastic volatility is almost an entire order higher.

In the same one-year ATM call example, we showed that under Black-Scholes, the option premium was 7.97% of the spot, which meant that a standard deviation in the PnL is 0.36% of the spot. Using the stochastic volatility quantity, we see that in the real case, the standard deviation of the PnL is actually around $0.35 \times 0.0797 \approx 0.0279$ or 2.79% of the spot. This is again, almost an entire order greater in magnitude than Black-Scholes.

2 Local Volatility

2.1 Black-Scholes to Dupire's Local Volatility

Under the risk-neutral measure $\mathbb{Q}$, we assume the follow Black-Scholes stock price dynamics,

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t^{\mathbb{Q}}.$$

From this, taking the log gives us the log-normal dynamics of the underlying stock price,

$$\begin{aligned}
d\log S_t &= \frac{1}{S_t}dS_t - \frac{1}{2}\frac{d\langle S\rangle_t}{S_t^2} \\
&= \frac{dS_t}{S_t} - \frac{d\langle S\rangle_t}{2S_t^2} \\
&= (r-q)dt + \sigma dW_t^{\mathbb{Q}} - \frac{1}{2}\sigma^2 dt \\
&= \left\{r - q - \frac{1}{2}\sigma^2\right\} dt + \sigma dW_t^{\mathbb{Q}}.
\end{aligned}$$

In explicitly form, this is simply,

$$\begin{aligned}
\log S_T &= \log S_0 + \left\{r - q - \frac{1}{2}\sigma^2\right\} T + \sigma W_T^{\mathbb{Q}} \\
S_T &= S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma W_T^{\mathbb{Q}}\right\} \\
&= S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}Z\right\},
\end{aligned}$$

since under Normality assumption $W_T^{\mathbb{Q}} = \sqrt{T}Z$ where $Z \sim \mathcal{N}(0,1)$. $W_T^{\mathbb{Q}}$ is a Brownian motion.

Under risk-neutral measure, the price of a European call option is the discounted expectation under the risk-neutral measure of the option's payoff. In this case, the payoff of a European call option is $(S_T - K)^+$ so we derive algebraically,

$$\begin{aligned}
C^{\text{BS}}(S_0, K, T, \sigma) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)\mathbf{1}\{S_T > K\}\right] \\
&= e^{-rT}\left\{\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbf{1}\{S_T > K\}\right] - K\mathbb{Q}(S_T > K)\right\}.
\end{aligned}$$

We first wish to make sense of $\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbf{1}\{S_T > K\}\right]$. Substituting the explicit form of S_T we can derive an explicit form for the comparison $S_T > K$,

$$\begin{aligned}
S_T &> K \\
S_0 \exp\left\{\left(r - q - \frac{1}{2}\right)\sigma^2 + \sigma\sqrt{T}Z\right\} &> K \\
\log S_0 + \left(r - q - \frac{1}{2}\right) T + \sigma\sqrt{T}Z &> \log K \\
\frac{\log\left(\frac{K}{S_0}\right) - \left(r - q - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} &< Z \iff S_T > K
\end{aligned}$$

If we let $d_2 = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ then it follows that $S_T > K$ is equivalent to $Z > -d_2$ which means that $\mathbb{Q}(S_T > K) = \mathbb{Q}(Z > -d_2) = N(-d_2)$. Since we have an explicit form for $S_T > K$, we will explicitly derive the entire term. We will let $a = \sigma\sqrt{T}$ for the sake of algebraic convenience.

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z\right\}\mathbf{1}\left\{Z>-d_2\right\}\right] \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}\mathbb{E}\left[\exp\left\{\sigma\sqrt{T}Z\right\}\mathbf{1}\left\{Z>-d_2\right\}\right] \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}\int_{-d_2}^{\infty} e^{az}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{az-\frac{1}{2}z^2\right\}dz \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(z^2-2az\right)\right\}dz \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(z-a)^2+\frac{1}{2}a^2\right\}dz \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}e^{a^2/2}\int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(z-a)^2\right\}dz \\
&\quad = S_0 \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}e^{a^2/2}\int_{-d_2-a}^{\infty} \phi(u)du, \quad u := z-a \implies z=-d_2 \wedge u=-d_2-a, \\
&\quad = e^{a^2/2}N(d_2+a).
\end{aligned}
$$

Since we have d_2 then we can let d_1 be,

$$
d_1 = \frac{\log\left(\frac{S_0}{K}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)}{\sigma\sqrt{T}}+\sigma\sqrt{T} = \frac{\log\left(\frac{S_0}{K}\right)+\left(r-q-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = d_2+\sigma\sqrt{T},
$$

which is precisely the term inside $N(d_2+a)$. Therefore we have an explicit closed-form solution,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[S_T\mathbf{1}\left\{S_T>K\right\}\right] &= S_0\exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)T\right\}\exp\left\{\frac{1}{2}\sigma^2 T\right\}N(d_1) \\
&= S_0\exp\left\{(r-q)T\right\}N(d_1).
\end{aligned}
$$

Trivially, it follows that the Black-Scholes price for a European call option is,

$$
\begin{aligned}
C^{\mathrm{BS}}(S_0,K,T,\sigma) &= e^{-rT}\left[S_0\exp\left\{(r-q)T\right\}N(d_1)-KN(d_2)\right] \\
&= e^{-rT}e^{(r-q)T}S_0N(d_1)-Ke^{-rT}N(d_2) \\
&= S_0e^{-qT}N(d_1)-Ke^{-rT}N(d_2).
\end{aligned}
$$

We wish to calibrate the implied surface of the Black-Scholes European call option price to the market call option prices, or in other words, solve the equality for a $\sigma_{\mathrm{imp}}(K,T)$ such that,

$$
C^{\mathrm{BS}}(S_0,K,T,\sigma_{\mathrm{imp}}(K,T)) = C^{\mathrm{mkt}}(K,T).
$$

We will define the notation $c(0,T,x,K)$. Suppose $x=S_0$, then we can write the market call option surface as $c(0,T,x,K)=e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+ \mid x=S_0]$ representing the current market call option price with maturity T, at stock stock price x and strike K as the discounted risk-neutral expected payoff.

For an arbitrary differentiable function $g(Y)$ where Y is a random variable, the expectation of a function applied on a random variable can be written as,

$$
\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(y)\phi(y)dy,
$$

where $\phi(y)$ represents the density function of the random variable Y. $(S_T-K)^+ = \max(S_T-K,0)$ which is differentiable everywhere except at $S_T=K$ where a kink exists. Therefore, we write the risk-neutral expectation of the payoff function as,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+] &= \int_{-\infty}^{\infty}(S_T-K)^+\cdot\tilde{p}(0,T,x,y)dS_t \\
&= \int_{-\infty}^{\infty}(y-K)^+\cdot\tilde{p}(0,T,x,y)dy.
\end{aligned}
$$

Here we model the transition density as the probability density of where the stock ends up at time T conditioning on the fact it started at x at time 0. This removes the Black-Scholes log-normal dynamics and allows us to compute without loss of generality. Under measure theory, this is simply the probability that the stock moves from state x at time 0 to somewhere near y at time T,

$$\mathbb{P}(S_T \in [y, y+dy] \mid S_0 = x) \approx \tilde{p}(0, T, x, y)dy,$$

i.e., a conditioned density. So, replacing the expectation by an integral against the transition density,

$$\begin{aligned} c(0, T, x, K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+ \mid S_0 = x\right] \\ &= e^{-rT}\int_0^{\infty}(y-K)^+ \cdot \tilde{p}(0, T, x, y)dy. \end{aligned}$$

We aim to show how the whole option price surface evolves through the forward Kolmogorov equation, so we must find c_T, c_K, c_{KK} and combine them to solve explicitly for the unknown instantaneous variance $\sigma^2_{\text{loc}}(K, T)$. The K derivatives will recover the risk-neutral terminal density $c_{KK} = e^{-rT}p(T, K)$ while the T derivatives measure how the entire option price surface evolves with time. Recall the Kolmogorov equations first.

Suppose we have a Markov process X_t with transition density $p(s, x; t, y)$ meaning that,

$$p(s, x; t, y)dy = \mathbb{P}(X_t \in [y, y+dy] \mid X_s = x).$$

If the process is at state x at time s, then $p(s, x; t, y)$ describes the density of ending at state y at a later time t. The starting point is the Markov property together with the Chapman-Kolmogorov equation that is,

$$p(s, x; t, y) = \int p(s, x; u, z)p(u, z; t, y)dz, \qquad s < u < t.$$

That is, the probability going from (s, x) to (t, y) can be decomposed by conditioning on an intermediate state z at time u. Suppose that X_t follows a diffusion $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$. Suppose that $a(t, x) = \sigma^2(t, x)$ such that it's infinitesimal generator of the diffusion is,

$$\mathcal{L} = \mu(t, x)\frac{\partial}{\partial x} + \frac{1}{2}a(t, x)\frac{\partial^2}{\partial x^2}.$$

The operator above encodes the local drift and local variance of the process. Thus, the two Kolmogorov equations arise depending on which time variable we differentiate respect to. The *backward equation* evolves with respect to the starting time s solving the future transition probability change given I change the initial time or initial state. The *forward equation* evolves with respect to the terminal t, giving the evolution of the probability mass itself given that time moves forward. Refer to the appendix for these equations.

Returning to the risk-neutral discounted expected payoff of our European call option, we will derive c_K, c_{KK} and c_T and derive an explicit closed-form for σ^2_{loc} also known as *Dupire's Local Volatility Model.*

We currently have,

$$\begin{aligned} c(0, T, x, K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T - K)^+ \mid x = S_0] \\ &= e^{-rT}\int_K^{\infty}(y-K)\tilde{p}(0, T, x, y)dy. \end{aligned}$$

The derivative of c wrt c_K is computed as,

$$\begin{aligned} c_K(0, T, x, K) &= e^{-rT}\frac{\partial}{\partial K}\int_K^{\infty}(y-K)^+ \cdot \tilde{p}(0, T, x, y)dy \\ &= e^{-rT}\left\{\cancel{-(K-K)\cdot\tilde{p}(0, T, x, y)} + \int_K^{\infty}(-1)\tilde{p}(0, T, x, y)dy\right\} \\ &= -e^{-rT}\int_K^{\infty}\tilde{p}(0, T, x, y)dy. \end{aligned}$$

Taking the derivative of c_K wrt K again gives us c_{KK},

$$\begin{aligned}c_{KK}(0,T,x,K) &= -e^{-rT}\frac{\partial}{\partial K}\int_K^\infty \tilde{p}(0,T,x,y)dy \\ &= -e^{-rT}\left\{-\tilde{p}(0,T,x,K)\right\} \\ &= e^{-rT}\tilde{p}(0,T,x,K).\end{aligned}$$

Computing c_T is a bit more involved. First we can write c_T as,

$$\begin{aligned}c_T(0,T,x,K) &= \frac{\partial}{\partial T}\left\{e^{-rT}\int_K^\infty (y-K)^+\tilde{p}(0,T,x,y)dy\right\} \\ &= -re^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy + e^{-rT}\int_K^\infty (y-K)^+\cdot\tilde{p}_T(0,T,x,y)dy \\ &= -re^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy + e^{-rT}\int_K^\infty (y-K)\left[\underbrace{-\frac{\partial}{\partial y}((r-q)y\tilde{p})}_{(1)} + \underbrace{\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2_{\text{loc}}(T,y)y^2\tilde{p}\right)}_{(2)}\right]dy.\end{aligned}$$

Solving for component (1) in the square brackets above,

$$\begin{aligned}-\int_K^\infty (y-K)\frac{\partial}{\partial y}(r-q)y\tilde{p}dy &= -\left[(y-K)(r-q)y\tilde{p}\right]_{y=K}^\infty + \int_K^\infty (r-q)y\tilde{p}dy \\ &= \int_K^\infty (r-q)y\tilde{p}dy \\ &= (r-q)\int_K^\infty \left\{(y-K)+K\right\}\tilde{p}dy \\ &= (r-q)\int_K^\infty (y-K)\tilde{p}dy + (r-q)K\int_K^\infty \tilde{p}dy.\end{aligned}$$

The 2nd line follows because $y-K=0$ for $y=K$ and $y=\infty$ kills the tail density. Component (2) is then,

$$\begin{aligned}\frac{1}{2}\int_K^\infty (y-K)\left[\frac{\partial^2}{\partial y^2}\sigma^2_{\text{loc}}(T,y)y^2\tilde{p}\right]dy &= \frac{1}{2}\left[(y-K)\frac{\partial}{\partial y}\sigma^2_{\text{loc}}(T,y)y^2\tilde{p}\right]_{y=K}^\infty - \frac{1}{2}\int_K^\infty \sigma^2_{\text{loc}}(T,y)y^2\tilde{p}dy \\ &= -\frac{1}{2}\int_K^\infty \frac{\partial}{\partial y}\left(\sigma^2_{\text{loc}}(T,y)y^2\tilde{p}\right)dy \\ &= -\frac{1}{2}\left[\sigma^2_{\text{loc}}(T,y)y^2\tilde{p}\right]_{y=K}^\infty \\ &= \frac{1}{2}\sigma^2_{\text{loc}}(T,K)K^2\cdot\tilde{p}(0,T,x,K).\end{aligned}$$

Then, c_T therefore becomes,

$$\begin{aligned}c_T(0,T,x,K) &= -re^{-rT}\int_K^\infty (y-K)\tilde{p}\,dy + (r-q)e^{-rT}\int_K^\infty (y-K)\tilde{p}\,dy \\ &\quad + (r-q)Ke^{-rT}\int_K^\infty \tilde{p}\,dy \\ &\quad + \frac{1}{2}e^{-rT}\sigma^2_{\text{loc}}(T,K)\,K^2\,\tilde{p}(0,T,x,K) \\ &= -q\underbrace{e^{-rT}\int_K^\infty (y-K)\tilde{p}dy}_{=c} + (r-q)\underbrace{Ke^{-rT}\int_K^\infty \tilde{p}dy}_{=c_K} + \frac{1}{2}\sigma^2_{\text{loc}}(T,K)K^2\cdot\underbrace{e^{-rT}\tilde{p}(0,T,x,K)}_{=c_{KK}}.\end{aligned}$$

We can re-write this using short-form notation of those respective derivatives in the underbraces to get,

$$\begin{aligned}c_T = -qc-(r-q)Kc_K+\frac{1}{2}\sigma^2_{\text{loc}}(T,K)K^2c_{KK} &\iff \frac{1}{2}\sigma^2_{\text{loc}}(T,K)K^2c_{KK} = c_T+qc+(r-q)Kc_K \\ &\iff \sigma^2_{\text{loc}}(T,K) = \frac{2[c_T+qc+(r-q)Kc_K]}{K^2c_{KK}}.\end{aligned}$$

When the stock pays no dividend we have that our local volatility is,

$$\sigma^2_{\text{loc, no-div.}}(T,K) = \frac{2[c_T + rKc_K]}{K^2 c_{KK}}.$$

The full market call surface is specified when,

$$C^{\text{BS}}(S_0, K, T, \sigma_{\text{imp}}(K,T)) = C^{\text{mkt}}(K,T).$$

Substituting, we get *Dupire's Local Volatility* definition,

$$\sigma^2_{\text{loc}}(K,T) = \frac{2[\partial_T C^{\text{mkt}}(T,K) + qC^{\text{mkt}}(T,K) + (r-q)K\partial_K C^{\text{mkt}}(T,K)]}{K^2 \partial_{KK} C^{\text{mkt}}(T,K)}.$$

2.2 No-Arbitrage Conditions

In this section, I aim to identify where Dupire's formulation is well-defined and arbitrage-free. Well-defined and arbitrage-free in the option's volatility modelling space often go hand in hand.

Previously we had derived the local volatility formula as,

$$\sigma^2_{\text{loc}} = \frac{2[c_T + qc + (r-q)Kc_K]}{K^2 c_{KK}}.$$

Trivially, we must have that $c_{KK} > 0$ for all strikes. But the less obvious arbitrage statements we wish to justify is why the denominator and numerator must be positive.

2.2.1 Strike-Arbitrage

Trivially, since $K^2 > 0$ we require that $c_{KK}(T,K) \geq 0$. Consequently, this says that the call price must be convex as a function of the strike. We show this by approximating the second derivative by the symmetric finite difference given by the limit,

$$\frac{\partial^2 c}{\partial K^2}(K,T) = \lim_{\varepsilon \to 0} \frac{c(K-\varepsilon,T) - 2c(K,T) + c(K+\varepsilon,T)}{\varepsilon^2}.$$

The numerator is clearly the price of the portfolio given by,

$$\frac{1}{\varepsilon^2}C(K-\varepsilon,T) - \frac{2}{\varepsilon^2}C(K,T) + \frac{1}{\varepsilon^2}C(K+\varepsilon,T).$$

Thus, we'll write the terminal payoff as,

$$B_\varepsilon(S_T) = \frac{1}{\varepsilon^2}\left[(S_T - (K-\varepsilon))^+ - 2(S_T - K)^+ + (S_T - (K+\varepsilon))^+\right].$$

I'll evaluate $B_\varepsilon(S_T)$ on each of the 4 regions it is defined on.

If $S_T \leq K - \varepsilon$, all three calls are OTM so,

$$B_\varepsilon(S_T) = 0.$$

If $K - \varepsilon < S_T \leq K$, only the $K - \varepsilon$ call is ITM so,

$$B_\varepsilon(S_T) = \frac{1}{\varepsilon^2}[S_T - K + \varepsilon] = \frac{S_T - K + \varepsilon}{\varepsilon^2}.$$

This is clearly a positive linearly rising function from 0 to $\frac{1}{\varepsilon}$.

If $S_T \geq K + \varepsilon$, all three calls are ITM so,

$$\begin{aligned} B_\varepsilon(S_T) &= \frac{1}{\varepsilon^2}\left[(S_T - K + \varepsilon) - 2(S_T - K) + (S_T - K - \varepsilon)\right] \\ &= \frac{1}{\varepsilon^2}[S_T - K + \varepsilon - 2S_T + 2K + S_T - K - \varepsilon] \\ &= 0. \end{aligned}$$

It follows that the terminal payoff function is a triangular spike centered at K,

$$B_\varepsilon(S_T) = \begin{cases} 0, & S_T \leq K - \varepsilon, \\ \dfrac{S_T - K + \varepsilon}{\varepsilon^2}, & K - \varepsilon < S_T \leq K, \\ \dfrac{K + \varepsilon - S_T}{\varepsilon^2}, & K < S_T < K + \varepsilon, \\ 0, & S_T \geq K + \varepsilon. \end{cases}$$

It has height $\frac{\varepsilon}{\varepsilon^2} = \frac{1}{\varepsilon^2}$ with base width 2ε and total area equal to $\frac{1}{2}(2\varepsilon) \cdot \frac{1}{\varepsilon} = 1$.

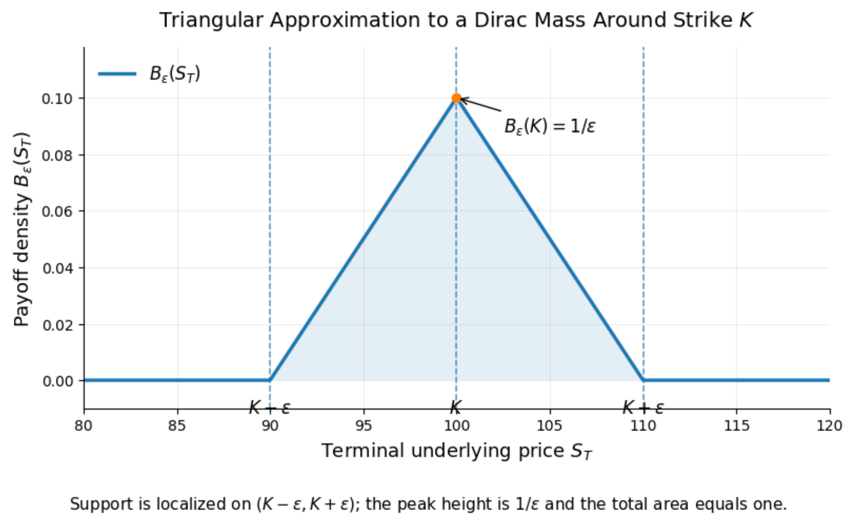

Figure 1: Triangular approximation to a Dirac mass around the strike K.

As $\varepsilon \to 0$ this payoff behaves like a Dirac delta centered at K. Moreover, it is never negative, meaning that the option price can never be negative. Therefore we conclude that

$$\frac{c(K - \varepsilon, T) - 2c(K, T) + c(K + \varepsilon, T)}{\varepsilon^2} \geq 0,$$

and taking thre limit gives that $c_{KK}(T, K) \geq 0$. We can also see this from the density,

$$c(T, K) = e^{-rT}\mathbb{E}[(S_T - K)^+].$$

Taking the first derivative with respect to strike we have,

$$\begin{aligned} c_K(T, K) &= e^{-rT}\mathbb{E}\left[\frac{\partial}{\partial K}(S_T - K)^+\right] \\ &= e^{-rT}\mathbb{E}[-\mathbf{1}\{S_T > K\}] \\ &= -e^{-rT}\mathbb{Q}(S_T > K). \end{aligned}$$

Taking the derivative of the first derivative wrt to K again,

$$\begin{aligned} c_{KK}(T,K) &= -e^{-rT}\frac{\partial}{\partial K}\mathbb{Q}(S_T > K) \\ &= -e^{-rT}\frac{\partial}{\partial K}\int_K^\infty p_T(y)dy \\ &= -e^{-rT}(-p_T(K)) = e^{-rT}p_T(K) \geq 0. \end{aligned}$$

Clearly, c_{KK} is the discounted risk-neutral density at strike K. So, if $c_{KK} < 0$, the market is assigning a negative density to some terminal stock level which is impossible and creates butterfly arbitrage.

2.2.2 Maturity Arbitrage

The numerator of our local volatility equation contains the component $c_T + qc + (r-q)Kc_K$. To see the maturity-arbitrage condition more clearly, we can re-write this as a maturity derivative of a forward-moneyness-normalized call-option price. Recall that the forward price at time 0 to maturity T with fixed proportional strike for $k > 0$ is written as $F_T = xe^{(r-q)T}$. The actual strike at maturity T is written as,

$$K(T) = kF_T.$$

We also define $h(T) = e^{qT}c(T, K(T))$ and differentiate $h(T)$ using the product and chain rule,

$$\begin{aligned} h'(T) &= \frac{d}{dT}\left[e^{qT}c(T,K(T))\right] \\ &= qe^{qT}c(T,K(T)) + e^{qT}\frac{d}{dT}c(T,K(T)) \\ &= qe^{qT}c(T,K(T)) + e^{qT}\left[c_T(T,K(T)) + c_K(T,K(T))K'(T)\right] \\ &= e^{qT}\left[qc(T,K(T)) + c_T(T,K(T)) + c_K(T,K(T))K'(T)\right] \\ &= e^{qT}\left[qc + c_T + (r-q)Kc_K\right]. \end{aligned}$$

This follows because $K(T) = kF_T = kxe^{(r-q)T}$ so $K'(T) = (r-q)kxe^{(r-q)T} = (r-q)K(T)$ by definition. Hence, the numerator is positive if and only if $T \mapsto e^{qT}c(T, kF_T)$ is increasing for each fixed strike k. Explicitly,

$$c_T + qc + (r-q)Kc_K = e^{-qT}\frac{d}{dT}\left[e^{qT}c(T,kF_T)\right].$$

In the forward pricing sense, if we compare options at the same forward moneyness that is $k = K/F_T$, then the appropriately adjusted call price should increase with maturity. For $T_1 < T_2$ the no-arbitrage condition is,

$$e^{qT_1}c(T_1, kF_{T_1}) \leq e^{qT_2}c(T_2, kF_{T_2}).$$

Suppose for a second that this condition is violated. Then that means there exists $T_1 < T_2$ and k such that,

$$e^{qT_1}c(T_1, kF_{T_1}) > e^{qT_2}c(T_2, kF_{T_2}) \iff c(T_2, kF_{T_2}) < e^{-q(T_2-T_1)}c(T_1, kF_{T_1}).$$

So if we buy 1 call with maturity T_2 and strike kF_{T_2}, we incure a negative initial cost meaning we receive money at inception. By selling $e^{-q(T_2-T_1)}$ calls with maturity T_1 and strike kF_{T_1} gives an initial cost of,

$$c(T_2, kF_{T_2}) - e^{-q(T_2-T_1)}c(T_1, kF_{T_1}) < 0,$$

which is negative by the violated inequality. Thus our portfolio is constructed by being long a call with maturity T_2 and short a call with maturity T_1. The payoffs of the long and short maturity calls are respectively,

$$(S_{T_2} - kF_{T_2})^+, \qquad e^{-q\Delta T}(S_{T_1} - kF_{T_1})^+.$$

We must finance to time T_2 so the payoff becomes,

$$e^{r\Delta T}e^{-q\Delta T}(S_{T_1} - kF_{T_1})^+ = e^{(r-q)\Delta T}(S_{T_1} - kF_{T_1})^+.$$

In terms of forward pricing, the financing of the payoff above can be written as,

$$\frac{F_{T_2}}{F_{T_1}} = \frac{xe^{(r-q)T_2}}{xe^{(r-q)T_1}} = e^{(r-q)(T_2-T_1)} = e^{(r-q)\Delta T}.$$

We also enter a delta position in the stock, that is, $\Delta = 0$ if the call option is OTM and (-1) if the option is ITM. Therefore, the self-financing PnL from holding Δ shares from T_1 to T_2 after financing and dividends becomes,

$$\Delta\left\{\frac{F_{T_2}}{F_{T_1}}(S_{T_1} - kF_{T_1})^+\right\} = \Delta\left(S_{T_2} - \frac{F_{T_2}}{F_{T_1}}S_{T_1}\right).$$

Therefore the total PnL at time T_2 is given by,

$$\Pi_{T_2} = (S_{T_2} - kF_{T_2})^+ - \frac{F_{T_2}}{F_{T_1}}(S_{T_1} - kF_{T_1})^+ + \Delta\left(S_{T_2} - \frac{F_{T_2}}{F_{T_1}}S_{T_1}\right).$$

Suppose we also have a forward-scaled T_1 stock valued at $S^*_{T_1} = \frac{F_{T_2}}{F_{T_1}}S_{T_1}$ which means that $\frac{F_{T_2}}{F_{T_1}}kF_{T_1} = kF_{T_2}$. So naturally, the payoff of the financed payoff of the short-term maturity T_1 call is,

$$\frac{F_{T_2}}{F_{T_1}}(S_{T_1} - kF_{T_1})^+ = (S^*_{T_1} - kF_{T_2})^+.$$

So it is the case that $S_{T_1} > kF_{T_1}$ is equivalent to $S^*_{T_1} > kF_{T_2}$.

Therefore our delta is $-\mathbf{1}\{S^*_{T_1} > kF_{T_2}\}$ so the PnL of our position is,

$$\Pi_{T_2} = \underbrace{(S_{T_2} - kF_{T_2})^+}_{=f(S_{T_2})} - \underbrace{(S^*_{T_1} - kF_{T_2})^+}_{=f(S^*_{T_1})} - \underbrace{\mathbf{1}\left\{S^*_{T_1}kF_{T_2}\right\}}_{=f'(S^*_{T_1})}(S_{T_2} - S^*_{T_1}),$$

if we let $f(x) = (x - kF_{T_2})^+$. Clearly this is a convex function, by definition. That is, for any convex function, $f(y) \geq f(x) + f'(x)(y - x)$ by *Supporting-Tangent Inequality*. Thus we get that,

$$f(S_{T_2}) \geq f(S^*_{T_2}) + f'(S^*_{T_1})(S_{T_2} - S^*_{T_1}).$$

If the maturity monotonicity condition is violated, we receive money at inception and still end with a non-negative terminal PnL. This is clearly maturity arbitrage.

More specifically, under risk-neutral measure with dividend yield,

$$\mathbb{E}^{\mathbb{Q}}[S_{T_2} \mid S_{T_1}] = S_{T_1}e^{(r-q)(T_2-T_1)} = \frac{F_{T_2}}{F_{T_1}}S_{T_1}.$$

Defining $f(x) = (x - kF_{T_2})^+$, by Jensen's inequality, since f is convex, we have that,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[f(S_{T_2}) \mid S_{T_1}] &\geq f\left(\mathbb{E}^{\mathbb{Q}}[S_{T_2} \mid S_{T_1}]\right) \\
&= f\left(\frac{F_{T_2}}{F_{T_1}}S_{T_1}\right) \\
&= \left(\frac{F_{T_2}}{F_{T_1}} - kF_{T_2}\right)^+ \\
&= \left(\frac{F_{T_2}}{F_{T_1}}(S_{T_1} - kF_{T_1})\right)^+ \\
&= \frac{F_{T_2}}{F_{T_1}}(S_{T_1} - kF_{T_2})^+.
\end{aligned}$$

Taking unconditional expectations we get,

$$\mathbb{E}^{\mathbb{Q}}\left[(S_{T_2}-kF_{T_2})^+\right] \geq \frac{F_{T_2}}{F_{T_1}}\mathbb{E}^{\mathbb{Q}}\left[(S_{T_1}-kF_{T_1})^+\right].$$

Multiplying both sides by $e^{-(r-q)T_2}$ we see that,

$$\begin{aligned} e^{-(r-q)T_2}\mathbb{E}^{\mathbb{Q}}\left[(S_{T_2}-kF_{T_2})^+\right] &\geq e^{-(r-q)T_2}e^{(r-q)(T_2-T_1)}\mathbb{E}^{\mathbb{Q}}\left[(S_{T_1}-kF_{T_1})^+\right] \\ &= e^{-(r-q)T_1}\mathbb{E}^{\mathbb{Q}}\left[(S_{T_1}-kF_{T_1})^+\right]. \end{aligned}$$

However, observe that,

$$\begin{aligned} e^{qT}c(T,kF_T) &= e^{qT}e^{-rT}\mathbb{E}^{\mathbb{Q}}[(S_T-kF_T)^+] \\ &= e^{-(r-q)T}\mathbb{E}^{\mathbb{Q}}\left[(S_T-kF_T)^+\right]. \end{aligned}$$

Therefore it must be the case that,

$$e^{qT_2}c(T_2,kF_{T_2}) \geq e^{qT_1}c(T_1,kF_{T_1}).$$

Therefore, in any arbitrage-free model, the maturity-adjusted price at fixed forward moneyness must increase with maturity. Therefore, the full no-arbitrage interpretation of Dupire's local volatility model is only well-defined when $c_{KK} \geq 0$ and when $c_T + qc + (r-q)Kc_K \geq 0$.

A butterfly spread cannot have negative price and non-negative payoff. Moreover, adjusted call prices must increase with maturity given a fixed forward-moneyness because then we can simply buy the longer option and sell the shorter option, delta-adjust the short maturity and lock in an arbitrage profit.

2.2.3 Convex-Order Condition for Implied Volatilities

Suppose again we fixed forward-moneyness, that is fix $K = kF_T$ for $F_T = S_0e^{(r-q)T}$.

Let us also define *total implied variance* as,

$$\Sigma(K,T) = \sigma_{\text{imp}}(K,T)\Big|_{K=kF_T}, \quad w(K,T) = T\Sigma^2(K,T).$$

It follows from the Black-Scholes formula for a call option price that,

$$\begin{aligned} c(T,K) &= S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2) \\ &= S_0e^{-qT}N(d_1) - kS_0e^{(r-q)T}e^{-rT}N(d_2), \qquad K = kF_T = kS_0e^{(r-q)T} \\ &= S_0e^{-qT}\left\{N(d_1) - KN(d_2)\right\}. \end{aligned}$$

Multiplying both sides by e^{qT} we get,

$$\begin{aligned} e^{qT}c(T,K) &= S_0\left[N(d_1) - KN(d_2)\right] \\ e^{qT}c(T,kF_T) &= S_0B(k,w(K,T)) \\ &:= S_0\left\{N\left(\frac{-\log k + \frac{1}{2}w}{\sqrt{w}}\right) - KN\left(\frac{-\log k - \frac{1}{2}w}{\sqrt{w}}\right)\right\}. \end{aligned}$$

We'll define,

$$B(k,w(K,T)) := \left\{N\left(\frac{-\log k + \frac{1}{2}w}{\sqrt{w}}\right) - KN\left(\frac{-\log k - \frac{1}{2}w}{\sqrt{w}}\right)\right\}.$$

What is left is to express the factors d_1 and d_2 in terms of total implied variance.

The algebra follows directly,

$$\begin{aligned} d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + \left(r - q + \frac{1}{2}\Sigma^2\right)T}{\Sigma\sqrt{T}} \\ &= \frac{-\log K + \frac{1}{2}\Sigma^2 T}{\Sigma\sqrt{T}} \\ &= \frac{-\log K + \frac{1}{2}w}{\sqrt{w}}. \end{aligned}$$

Note that the moneyness term in the numerator simplies after substituting,

$$\log\frac{S_0}{K} + (r-q)T = \log\left\{\frac{S_0}{KS_0(r-q)T}\right\} + (r-q)T = -\log K - (r-q)T + (r-q)T = -\log K.$$

Then, d_2 follows through the identity,

$$d_1 = d_2 + \Sigma^2\sqrt{T} \implies d_2 = d_1 - \Sigma^2\sqrt{w} = \frac{-\log k - \frac{1}{2}w}{\sqrt{w}}.$$

Therefore the convexity condition on implied variances is equivalent,

$$\begin{aligned} e^{qT_2}c(T_2, KF_{T_2}) \geq e^{qT_1}c(T_2, KF_{T_1}) &\iff S_0B(k, w(K,T_2)) \geq S_0B(k, w(K,T_1)) \\ &\iff B(k, w(K,T_2)) \geq B(k, w(K,T_1)). \end{aligned}$$

Let $m = \log k$ then,

$$d_1 - d_2 = \sqrt{w}, \quad d_1 + d_2 = -\frac{2m}{\sqrt{w}}.$$

The difference in squares of the factors is therefore,

$$d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2) = \sqrt{w} \cdot \left(-\frac{2m}{\sqrt{w}}\right) = -2m.$$

Using the normal cdf we get that,

$$\frac{\phi(d_1)}{\phi(d_2)} = \frac{e^{-d_1^2/2}}{e^{-d_2^2/2}} = e^{-(-2m)/2} = e^m = k.$$

Taking the derivative of B wrt w we get,

$$\begin{aligned} \frac{\partial B}{\partial w} &= \phi(d_1) \cdot \frac{\partial d_1}{\partial w} - K\phi(d_2) \cdot \frac{\partial d_2}{\partial w} \\ &= \phi(d_1) \cdot \left(\frac{\partial d_1}{\partial w} - \frac{\partial d_2}{\partial w}\right) \\ &= \phi(d_1) \cdot \frac{\partial}{\partial w} \cdot (d_1 - d_2) \\ &= \phi(d_1) \cdot \frac{\partial}{\partial w}\sqrt{w} \\ &= \frac{1}{2\sqrt{w}} \cdot \phi(d_1), \end{aligned}$$

which is always positive since $w > 0$ and $\phi(d_1) > 0$.

We conclude that $B(K, w)$ is a strictly increasing function in total variance.

More explicitly, it follows that,

$$
\begin{aligned}
B(K, w(K, T_2)) &\geq B(K, w(K, T_1)) \\
w(K, T_2) &= w(K, T_1) \\
T_2 \Sigma^2(K, T_2) &\geq T_1 \Sigma^2(K, T_1).
\end{aligned}
$$

Solving for $\Sigma(K, T_2)$ we get, that,

$$
\Sigma(K, T)_2 \geq \sqrt{\frac{T_1}{T_2}} \Sigma^2(K, T_1).
$$

2.3 Implied Volatilities to Local Volatilities

As we saw previously, it is often easier to view the strike in terms of forward log-moneyness, that is, to write the strike explicitly as a forward price level. We can therefore express K as $K = kF_T = kS_0 e^{(r-q)T}$.

Therefore, the call price under Black-Scholes can be written as,

$$
\begin{aligned}
C(T, K) &= S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2) \\
&= S_0 e^{-qT} N(d_1) - K S_0 e^{(r-q)T} e^{-rT} N(d_2) \\
&= S_0 \left\{ e^{-qT} N(d_1) - K e^{(\cancel{r}-q)T} \cancel{e^{-rT}} N(d_2) \right\} \\
&= S_0 e^{-qT} \left\{ N(d_1) - K N(d_2) \right\}.
\end{aligned}
$$

We'll let the the initial price term involving the exponent be $A(T)$ and let the difference of the normal CDF be a function of log forward-moneyness, that is $B(y, f(T, y))$. $f(T, y)$ will be defined as the total implied variance, that is $f(T, y) = T\hat{\sigma}^2(K, T)$ and $A(T) = S_0 e^{-qT}$ and $B(y, f(T, y)) = N(d_1) - KN(d_2)$.

Recall that Dupire's model of Local Volatility can be written as,

$$
\sigma^2_{\text{loc}} = \frac{2[c_T + qc + (r-q)Kc_K]}{K^2 c_{KK}}.
$$

Thus we solve explicitly for the derivatives c_T, c_K, c_{KK}. The derivatives of y are as follows,

$$
\frac{\partial y}{\partial y} = -(r-q), \quad \frac{\partial y}{\partial K} = \frac{1}{K},
$$

for a fixed strike K and fixed maturity T. c_T follows directly from the chain rule,

$$
\begin{aligned}
c_T &= A_T B + A \left[B_y y_T + B_f (f_T + f_y y_T) \right] \\
&= -qAB + A \left\{ -(r-q) B_y + B_f f_T - (r-q) B_f f_y \right\} \\
&= A \left\{ -qB + B_f f_T - (r-q) \left(B_f + B_f f_y \right) \right\}.
\end{aligned}
$$

We will keep this formulae for reference. c_K, again, follows directly from an application of chain rule,

$$
\begin{aligned}
c_K &= A \left\{ B_y + B_f f_y \right\} \cdot y_K \\
&= A \left\{ B_y + B_f f_y \right\} \frac{1}{K} \\
K c_K &= A \left\{ B_y + B_f f_y \right\} \\
\iff (r-q) K c_K &= (r-q) A \left\{ B_y + B_f f_y \right\}.
\end{aligned}
$$

Substituting these derivatives back into the numerator of Dupire's local volatility,

$$
\begin{aligned}
c_T + qC + (r-q)Kc_K &= A \left\{ \cancel{-qB} + B_f f_T - \cancel{(r-q)(B_y + B_f f_y)} \right\} + \cancel{qAB} + \cancel{(r-q)A(B_y + B_f f_y)} \\
&= AB_f f_T \\
&= S_0 e^{-qT} B_T f_T.
\end{aligned}
$$

For algebraic convenience, we will write $G := B_y + B_f f_y$ so that c_K becomes $c_K = A \cdot \frac{G}{K}$.

Then $c_K K$ follows directly from chain rule,

$$\begin{aligned} c_{KK} &= A \cdot \frac{\partial}{\partial K}\left(\frac{G}{K}\right) \\ &= A\left\{\frac{G_y}{K^2} - \frac{G}{K^2}\right\} \\ &= \frac{A}{K^2}\left\{G_y - G\right\} \\ \implies K^2 c_{KK} &= A\left\{G_y - G\right\}. \end{aligned}$$

G_y follows from another application of chain rule,

$$\begin{aligned} G_y &= B_{yy} + \frac{\partial}{\partial y}(B_f f_y) \\ &= B_{yy} + B_{yf} f_y + B_{ff} f_y^2 + B_f f_{yy} + B_{fy} f_y \\ &= B_{yy} + 2B_{yf} f_y + B_{ff} f_y^2 + B_f f_{yy}. \end{aligned}$$

The derivatives B_y and B_{yy} follow as,

$$\begin{aligned} B_y &= \phi(d_1) d_{1,y} - e^y N(d_2) - e^y \phi(d_2) d_{2,y} \\ &= -\frac{\phi(d_1)}{\sqrt{f}} - e^y N(d_2) + \frac{e^y \phi(d_2)}{\sqrt{f}} \\ &= \phi(d_1) d_{1,f} - e^y \phi(d_2) d_{2,f} \\ &= \phi(d_1)(d_{1,f} - d_{2,f}) \\ &= \frac{1}{2\sqrt{f}}\phi(d_1) \\ B_{yy} &= -e^y N(d_2) - e^y \phi(d_2) d_{2,y} \\ &= B_y - e^y \phi(d_2)\left(-\frac{1}{\sqrt{f}}\right) \\ &= B_y + \frac{e^y \phi(d_2)}{\sqrt{f}} \\ &= B_y + \phi(d_1) \cdot \frac{1}{\sqrt{f}} \\ &= B_y + 2B_f. \end{aligned}$$

Therefore $B_{yy} - B_y = 2B_f$. We will require the differences,

$$d_2 = -yf^{1/2} - \frac{1}{2}f^{1/2}, \quad d_{2,f} = \frac{y}{2}f^{-3/2} - \frac{1}{4}f^{-1/2} = \frac{y}{2y^{3/2}} - \frac{1}{4\sqrt{f}}.$$

Then it follows that B_{yf} is,

$$\begin{aligned} B_{yf} &= -\phi(d_1)\left\{\frac{y}{2f^{3/2}} - \frac{1}{4\sqrt{f}}\right\} \\ &= \phi(d_1)\left\{\frac{1}{4\sqrt{f}} - \frac{y}{2f^{3/2}}\right\} \\ &= \frac{\phi(d_1)}{2\sqrt{f}}\left\{\frac{1}{2} - \frac{y}{f}\right\} \\ &= B_f\left\{\frac{1}{2} - \frac{y}{f}\right\}. \end{aligned}$$

Therefore the difference term becomes,

$$2B_{yf} - B_F = 2B_f\left\{\frac{1}{2} - \frac{y}{f}\right\} - B_y = B_f - \frac{2y}{f}B_f - B_f = -\frac{2y}{f}B_f.$$

What remains is to determine what B_{ff} is in closed-form.

Recall that we had that $B_f = \frac{\phi(d_1)}{2\sqrt{f}}$ and we can write the log of B_f as,

$$\log B_f = \log \phi(d_1) - \log 2 - \frac{1}{2}\log f.$$

To simplify the cancellations a bit we will write the derivatives as a ratio,

$$\begin{aligned}
\frac{B_{ff}}{B_f} &= \frac{\partial}{\partial f}\log B_f \\
&= \frac{\partial}{\partial f}\log \phi(d_1) - \frac{1}{2f} \\
&= -d_1 d_{1,f} - \frac{1}{2f} \\
&= -\left\{-\frac{y}{\sqrt{f}} + \frac{1}{2}\sqrt{f}\right\}\cdot\left\{\frac{y}{2f^{3/2}} + \frac{1}{4\sqrt{f}}\right\} - \frac{1}{2f} \\
&= -\left(\frac{y^2}{2f^2} - \cancel{\frac{y}{4f}} + \cancel{\frac{y}{4f}} + \frac{1}{8}\right) - \frac{1}{2f} \\
&= -\left\{-\frac{y^2}{2f^2} + \frac{1}{8}\right\} - \frac{1}{2f} \\
&= \frac{y^2}{2f^2} - \frac{1}{2f} - \frac{1}{8} \\
&= \frac{1}{2}\left(\frac{y^2}{f^2} - \frac{1}{f} - \frac{1}{4}\right).
\end{aligned}$$

Substituting this into the difference $G_y - G$,

$$\begin{aligned}
G_y - G &= 2B_f - \frac{2y}{f}B_f f_y + \frac{1}{2}B_f\left\{\frac{y^2}{f^2} - \frac{1}{f} - \frac{1}{4}\right\} f_y^2 + B_f f_{yy} \\
&= 2B_f\left\{1 - \frac{y}{f}f_y + \frac{1}{4}\left(\frac{y^2}{f^2} - \frac{1}{f} - \frac{1}{4}\right) f_y^2 + \frac{1}{2}f_{yy}\right\}.
\end{aligned}$$

Re-arranging for squared strike scaled by convexity with respect to strike we get,

$$K^2 c_{KK}. = S_0 e^{-qT}\cdot 2B_f\left\{1 - \frac{y}{f}f_y + \frac{1}{2}f_{yy} + \frac{1}{4}\left(\frac{y^2}{f^2} - \frac{1}{f} - \frac{1}{4}f_y^2\right)\right\}$$

Substituting our definitions we get,

$$\begin{aligned}
\sigma^2_{\text{loc}}(T,K) &= \frac{2S_0 e^{-qT} B_f f_T}{S_0 e^{-qT}\cdot 2B_f\left[1 - \frac{y}{f}f_y + \frac{1}{2}f_{yy} + \frac{1}{4}\left(\frac{y^2}{f^2} - \frac{1}{f} - \frac{1}{4}\right) f_y^2\right]} \\
&= \frac{f_T}{1 - \frac{y}{f}f_y + \frac{1}{2}f_{yy} + \frac{1}{4}\left(\frac{y^2}{f^2} - \frac{1}{f} - \frac{1}{4}\right) f_y^2} \\
&= \frac{f_T}{1 - \frac{y}{f}f_y + \frac{y^2}{4f^2}f_y^2 + \frac{1}{2}f_{yy} - \frac{1}{4}\left(\frac{1}{f} + \frac{1}{4}\right) f_y^2} \\
&= \left.\frac{f_T}{\left(1 - \frac{y}{2f}f_y\right)^2 + \frac{1}{2}f_{yy} - \frac{1}{4}\left(\frac{1}{f} + \frac{1}{4}\right) f_y^2}\right|_{y=\log(K/F_T)}.
\end{aligned}$$

This formulation provides us with a direct mapping from the implied total variance surface to the local volatility surface. The numerator, in particular, is the maturity slope of total implied variance at a fixed forward log-moneyness, or in other words, the maturity-arbitrage condition where $f_T(T,y) \geq 0$. The denominator, evidently, is proportional to c_{KK} because $K^2 c_{KK} = S_0 e^{-qT}\cdot 2B_f$ scaled by some denominator.

Since $S_0 e^{-qT} > 0$ and $B_f > 0$, the denominator positive exactly when the call surface is convex in strike, that is, when there is no butterfly arbitrage. So, the implied implied-to-local vol formula is really just a Dupire local volatility model re-written in the coordinate (y, T, f) that is $(K, T, \hat{\sigma}) \mapsto (y, T, f)$ where $y = \log(K/F_T)$ and $f = T\hat{\sigma}^2$. Consider the flat volatility case as a sanity check.

Suppose that $\hat{\sigma}(K, T) = \sigma_0$ for all fixed strikes and maturities. Then $f(T, y) = \sigma_0^2 T$ so that $f_T = \sigma_0^2$ and $f_y = 0$ with $f_{yy} = 0$. The denominator is trivially just 1 so $\sigma_{\text{loc}}^2(T, S) = \sigma_0^2$.

Therefore, the flat implied volatility surface produces a flat local volatility surface, as it should.

In practice, we take market implied volatilities, convert them to total variances $f = T\hat{\sigma}^2$, express them as a smooth surface (T, y) and compute f_T, f_y, f_{yy} and the plug those derivatives into the formulation above. Empirically, this is the reason why the interpolation of the implied volatility surface matters, that is, the local volatility is not just using implied volatility levels but rather using their first maturity derviative, first moneyness derivative and second moneyness derivative.

2.4 Dynamics of Dupire's Local Volatility Model

Given that we have already expressed local volatilities as functions of implied volatilities, we aim to derive the dynamics of the local volatility model for a given local volatility function $\sigma(t, S)$, specifically, how local volatilities respond to a move in the underlying spot price.

The forward equation is often difficult to solve so we will use an approximation method that expresses the market implied volatilities as a function of the local volatility function directly.

Appendix

Kolmogorov Equations

Forward Kolmogorov Equation

$$
\begin{aligned}
\frac{\partial}{\partial t}p(s,x;t,y) &= \lim_{h\downarrow 0}\frac{p(s,x;t+h,y)-p(s,x;t,y)}{h} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left\{\int_{\mathbb{R}} p(s,x;t,z)p(t,z;t+h,y)\,dz - p(s,x;t,y)\right\} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left\{\int_{\mathbb{R}} p(s,x;t,z)\left[p(t,z;t+h,y)-\delta_y(z)\right]\,dz\right\} \\
&= \int_{\mathbb{R}} p(s,x;t,z)\,\lim_{h\downarrow 0}\frac{p(t,z;t+h,y)-\delta_y(z)}{h}\,dz \\
&= \int_{\mathbb{R}} p(s,x;t,z)\,\mathcal{L}_t^{*,y}\delta_y(z)\,dz \\
&= \mathcal{L}_t^{*,y}p(s,x;t,y).
\end{aligned}
$$

The *Forward Kolmogorov Equation* is therefore,

$$
\boxed{\frac{\partial}{\partial t}p(s,x;t,y) = \mathcal{L}_t^{*,y}p(s,x;t,y)}
$$

Backward Kolmogorov Equation

$$
\begin{aligned}
\frac{\partial}{\partial s}p(s,x;t,y) &= \lim_{h\downarrow 0}\frac{p(s+h,x;t,y)-p(s,x;t,y)}{h} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left\{p(s+h,x;t,y)-p(s,x;t,y)\right\} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left\{p(s+h,x;t,y)-\int_{\mathbb{R}} p(s,x;s+h,z)p(s+h,z;t,y)\,dz\right\} \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left\{\int_{\mathbb{R}} \delta_x(z)p(s+h,z;t,y)\,dz-\int_{\mathbb{R}} p(s,x;s+h,z)p(s+h,z;t,y)\,dz\right\} \\
&= \lim_{h\downarrow 0}\int_{\mathbb{R}}\frac{\delta_x(z)-p(s,x;s+h,z)}{h}p(s+h,z;t,y)\,dz \\
&= -\int_{\mathbb{R}}\lim_{h\downarrow 0}\frac{p(s,x;s+h,z)-\delta_x(z)}{h}p(s,z;t,y)\,dz \\
&= -\int_{\mathbb{R}}\mathcal{L}_s^{*,z}\delta_x(z)p(s,z;t,y)\,dz \\
&= -\int_{\mathbb{R}}\delta_x(z)\mathcal{L}_s^{z}p(s,z;t,y)\,dz \\
&= -\mathcal{L}_s^{x}p(s,x;t,y) \\
&= -\left[b(s,x)\frac{\partial}{\partial x}p(s,x;t,y)+\frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2}p(s,x;t,y)\right] \\
&= -b(s,x)\frac{\partial}{\partial x}p(s,x;t,y)-\frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2}p(s,x;t,y).
\end{aligned}
$$

The *Backward Kolmogorov Equation* is therefore,

$$
\boxed{\frac{\partial}{\partial s}p(s,x;t,y) = -b(s,x)\frac{\partial}{\partial x}p(s,x;t,y)-\frac{1}{2}\sigma^2(s,x)\frac{\partial^2}{\partial x^2}p(s,x;t,y)}
$$